Inhomogeneous approximation with coprime integers and lattice orbits

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ABSTRACT – Let (ξ, y) be a point in $\mathbb{R}^2$ and $\psi : \mathbb{N} \to \mathbb{R}^+$ a function. We investigate the problem of the existence of infinitely many pairs p, q of coprime integers such that

$$|q\xi + p - y| \le \psi(|q|).$$

We give both unconditional results which are valid for every real pair (ξ, y) with ξ irrational, and metrical results valid for almost all points (ξ, y). We link the subject with density exponents of lattice orbits in $\mathbb{R}^2$.

1 Introduction and results

Minkowski has proved that for every real irrational number ξ and every real number y not belonging to $\mathbb{Z}\xi + \mathbb{Z}$, there exist infinitely many pairs of integers p, q such that

$$|q\xi + p - y| \le \frac{1}{4|q|}.$$

See for instance Theorem II in Chapter 3 of Cassels' monograph [4]. The statement is optimal in the sense that the approximating function $\ell \mapsto (4\ell)^{-1}$ cannot be decreased. Note that the restriction $y \notin \mathbb{Z}\xi + \mathbb{Z}$ can be dropped at the cost of replacing the upper bound $(4|q|)^{-1}$ by $c|q|^{-1}$ for any constant c greater than $1/\sqrt{5}$. When $y = 0$, the primitive point $(\frac{p}{\gcd(p,q)}, \frac{q}{\gcd(p,q)})$ remains a solution to the above inequality, therefore we may moreover require that the pair of integers p, q be coprime. However, for a non-zero real number y, this extra requirement is far from being obvious to satisfy. In this direction, Chalk and Erdos [6] have obtained the following result:

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Theorem (Chalk-Erdos)**.** *Let ξ be an irrational real number and let y be a real number. There exists an absolute constant c such that the inequality*

$$|q\xi + p - y| \leq \frac{c(\log q)^2}{q(\log\log q)^2} \tag{1}$$

holds for infinitely many pairs of coprime integers (p, q) with q positive.

We study more generally the diophantine inequation

$$|q\xi + p - y| \leq \psi(|q|)$$

for coprime integers p and q, where $\psi : \mathbb{N} \to \mathbb{R}^+$ is a given function. Two types of questions naturally arise. First, finding unconditional results which are valid for every real pair (ξ, y) with ξ irrational as (1), and secondly getting metrical results valid for almost all points (ξ, y). Here is an example of the first kind.

Theorem 1. *Let ξ be an irrational real number and let y be a real number. There exist infinitely many integer quadruples (p_1, q_1, p_2, q_2) satisfying*

$$q_1 p_2 - p_1 q_2 = 1$$

and

$$|q_i\xi + p_i - y| \leq \frac{c}{\max(|q_1|, |q_2|)^{1/2}} \leq \frac{c}{\sqrt{|q_i|}}, \quad (i = 1, 2), \tag{2}$$

with $c = 2\sqrt{3}\max(1, |\xi|)^{1/2}|y|^{1/2}$.

Theorem 1 will be deduced in Section 2 from our results [10] of effective density for $SL(2, \mathbb{Z})$-orbits in $\mathbb{R}^2$. The estimate (2) is best possible, up to the value of the constant c. However, the optimality of (1) remains unclear. We address the following

Problem. *Can we replace the approximating function $\psi(\ell) = c(\log \ell)^2/\ell(\log\log \ell)^2$ occurring in (1) by a smaller one, possibly $\psi(\ell) = c\,\ell^{-1}$?*

We shall further discuss this problem in Section 4 for the function $\psi(\ell) = 2\,\ell^{-1}$, offering some hints and indicating the difficulties which then arise. It turns out that the approximating function $\psi(\ell) = \ell^{-1}$ is permitted for almost all pairs (ξ, y) of real numbers relatively to Lebesgue measure. The last assertion follows from the following metrical statement:

Theorem 2. *Let $\psi : \mathbb{N} \mapsto \mathbb{R}^+$ be a function. Assume that ψ is non-increasing, tends to 0 at infinity and that for every positive integer c there exists a positive real number c_1 satisfying*

$$\psi(c\,\ell) \geq c_1 \psi(\ell), \quad \forall \ell \geq 1. \tag{3}$$

Furthermore assume that

$$\sum_{\ell \geq 1} \psi(\ell) = +\infty.$$

Then, for almost all pairs (ξ, y) of real numbers there exist infinitely many primitive points (p, q) such that

$$q \geq 1 \quad \text{and} \quad |q\xi + p - y| \leq \psi(q). \tag{4}$$

If $\sum_{\ell \geq 1} \psi(\ell)$ converges, the pairs (ξ, y) satisfying (4) for infinitely many primitive points (p, q) form a set of null Lebesgue measure.

Note that we could have equivalently required in (4) that q be negative. Such a refinement could as well be achieved in the frame of Theorem 1, with a weaker approximating function of the form $\psi(\ell) = \ell^{-\mu}$ for any given real number $\mu < 1/3$, by employing alternatively Theorem 5 in Section 9 of [10]. We leave the details of proof to the interested reader, arguing as in Section 2. For questions of density involving signs, see also [7].

The proof of Theorem 2 is given in Section 3. It combines standard tools from metrical number theory with the ergodic properties of the linear action of $SL(2, \mathbb{Z})$ on $\mathbb{R}^2$ [13]. We refer to Harman's book [8] for closely related results. See also the recent overview [1] and the monographs [14], [15].

Theorem 2 is a metrical statement about pairs (ξ, y) of real numbers. A natural question is to understand what happens on each fiber when we fix either ξ or y. In this direction, here is a partial result which will be deduced from the explicit construction displayed in Section 4.

Theorem 3. *Let ξ be an irrational number and let $(p_k/q_k)_{k \geq 0}$ be the sequence of its convergents. Assume that the series*

$$\sum_{k \geq 0} \frac{1}{\max(1, \log q_k)} \tag{5}$$

diverges. Then for almost every real number y there exist infinitely many primitive points (p, q) satisfying

$$|q\xi + p - y| \leq \frac{2}{|q|}.$$

Moreover the series (5) diverges for almost every real number ξ.

We now turn to the second part of the paper devoted to density exponents for lattice orbits in $\mathbb{R}^2$. As already mentioned, the approximating function $\psi(\ell) = c\,\ell^{-1/2}$ occurring in Theorem 1 is directly connected to the density exponent $1/2$ for $SL(2,\mathbb{Z})$-orbits. We intend to show that this exponent $1/2$ is best possible in general.

We work in the more general setting of *lattices* Γ in $SL(2,\mathbb{R})$. Recall that a lattice Γ in $SL(2,\mathbb{R})$ is a discrete subgroup for which the quotient $\Gamma\backslash SL(2,\mathbb{R})$ has finite Haar measure. We view $\mathbb{R}^2$ as a space of column vectors on which the group of matrices Γ acts by left multiplication. We equip $\mathbb{R}^2$ with the supremum norm $|\ |$, and for any matrix $\gamma \in \Gamma$, we denote as well by $|\gamma|$ the maximum of the absolute values of the entries of γ. Let us first give a

Definition. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two points in* $\mathbb{R}^2$. *We denote by* $\mu_\Gamma(\mathbf{x},\mathbf{y})$ *the supremum, possibly infinite, of the exponents* μ *such that the inequality*

$$|\gamma\mathbf{x} - \mathbf{y}| \leq |\gamma|^{-\mu} \tag{6}$$

has infinitely many solutions $\gamma \in \Gamma$.

Note that for a fixed $\mathbf{x} \in \mathbb{R}^2$, the function $\mathbf{y} \mapsto \mu_\Gamma(\mathbf{x},\mathbf{y})$ is Γ-invariant. By the ergodicity of the action of Γ on $\mathbb{R}^2$, see [13], this function is therefore constant almost everywhere on $\mathbb{R}^2$. We denote by $\mu_\Gamma(\mathbf{x})$ its generic value and we call $\mu_\Gamma(\mathbf{x})$ the *generic density exponent* of the orbit $\Gamma\mathbf{x}$.

Theorem 4. *The upper bound* $\mu_\Gamma(\mathbf{x}) \leq 1/2$ *holds true for any point* $\mathbf{x} \in \mathbb{R}^2$ *such that the orbit* $\Gamma\mathbf{x}$ *is dense in* $\mathbb{R}^2$.

In an equivalent way, Theorem 4 asserts that the upper bound $\mu(\mathbf{x},\mathbf{y}) \leq 1/2$ holds for almost all points $\mathbf{y} \in \mathbb{R}^2$. This bound was already known in the case of the unimodular group $\Gamma = SL(2,\mathbb{Z})$ as a consequence of Theorem 3 in [10].

One may optimistically conjecture that $\mu_\Gamma(\mathbf{x}) = 1/2$ for every point $\mathbf{x}$ such that $\Gamma\mathbf{x}$ is dense in $\mathbb{R}^2$, or at least for almost every point $\mathbf{x} \in \mathbb{R}^2$. In this direction, it follows from [10] that the lower bound

$$\mu_{SL(2,\mathbb{Z})}(\mathbf{x}) \geq \frac{1}{3}$$

holds for all points $\mathbf{x}$ in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ with irrational slope. Weaker lower bounds can as well be deduced from [12] which are valid for any lattice $\Gamma \subset SL(2,\mathbb{R})$. Note that the function $\mathbf{x} \mapsto \mu_\Gamma(\mathbf{x})$ is Γ-invariant since the quantity $\mu_\Gamma(\mathbf{x})$ obviously depends only on the orbit $\Gamma\mathbf{x}$. Thus, the generic density exponent $\mu_\Gamma(\mathbf{x})$ takes the same value for almost all points $\mathbf{x} \in \mathbb{R}^2$.

Aknowledgement. We are grateful to Martin Windmer for calling our attention on the Chalk-Erdos Theorem in [6].

2 Proof of Theorem 1

We first state a result obtained in [10]. In this section, we denote by Γ the lattice $\mathrm{SL}(2,\mathbb{Z})$. For any point $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in $\mathbb{R}^2$ with irrational slope x_1/x_2, the orbit $\Gamma\mathbf{x}$ is dense in $\mathbb{R}^2$. We have obtained in [10] effective results concerning the density of such an orbit. In particular, our estimates are essentially optimal when the target point $\mathbf{y}$ has rational slope.

Lemma 1. *Let $\mathbf{x}$ be a point in $\mathbb{R}^2$ with irrational slope and $\mathbf{y} = \begin{pmatrix} y \\ y \end{pmatrix}$ a point on the diagonal with $y \neq 0$. Then, there exist infinitely many matrices $\gamma \in \Gamma$ such that*

$$|\gamma\mathbf{x} - \mathbf{y}| \leq \frac{c}{|\gamma|^{1/2}} \quad \text{with} \quad c = 2\sqrt{3}|\mathbf{x}|^{1/2}|y|^{1/2}. \tag{7}$$

Proof. The point $\mathbf{y}$ has rational slope 1. Apply Theorem 1 (ii) of [10] with $a = b = 1$. $\square$

Put $\mathbf{x} = \begin{pmatrix} \xi \\ 1 \end{pmatrix}$. The point $\mathbf{x}$ has irrational slope ξ so that Lemma 1 may be applied. Write $\gamma = \begin{pmatrix} q_1 & p_1 \\ q_2 & p_2 \end{pmatrix}$ a matrix provided by Lemma 1. Then, the inequality (7) gives

$$\begin{aligned} \max\left(|q_1\xi + p_1 - y|, |q_2\xi + p_2 - y|\right) &\leq \frac{c}{\max(|p_1|, |p_2|, |q_1|, |p_2|)^{1/2}} \\ &\leq \frac{c}{\max(|q_1|, |q_2|)^{1/2}}. \end{aligned}$$

Therefore, both points (p_1, q_1) and (p_2, q_2) satisfy (2), and since the determinant $q_1p_2 - q_2p_1 = 1$, the two integer points (p_1, q_1) and (p_2, q_2) are primitive. As there exist infinitely many matrices γ verifying (7), we thus find infinitely many solutions to (2).

Assume now that the irrational number ξ has bounded partial quotients. Then, Theorem 4 in [10] gives us in the opposite direction a lower bound of the form

$$|\gamma\mathbf{x} - \mathbf{y}| \geq \frac{c'}{|\gamma|^{1/2}},$$

for some positive constant c' depending only upon (ξ, y). Since $|\gamma| \leq c'' \max(|q_1|, |q_2|)$ when (2) holds, the estimate (2) is optimal up to the value of c.

Remark. The single inequality $|q_1\xi + p_1 - y| \leq \psi(|q_1|)$ geometrically means that the point $\gamma\mathbf{x}$ falls inside a neighborhood of the vertical line $x_1 = y$. A better understanding of the shrinking target problem for the dense orbit $\Gamma\mathbf{x}$, not to a point $\mathbf{y}$ as in [10] but to a line in $\mathbb{R}^2$, may possibly lead to a refinement of (1).

3 Proof of Theorem 2

It is convenient to view the pairs (ξ, y) occurring in Theorem 2 as column vectors $\begin{pmatrix} \xi \\ y \end{pmatrix}$ in $\mathbb{R}^2$. We are concerned with the set $\mathcal{E}(\psi)$ of vectors $\begin{pmatrix} \xi \\ y \end{pmatrix} \in \mathbb{R}^2$ for which there exist infinitely many primitive integer points (p, q) such that

$$q \geq 1 \quad \text{and} \quad |q\xi + p - y| \leq \psi(q). \tag{8}$$

For fixed p, q, denote by $\mathcal{E}_{p,q}(\psi)$ the strip

$$\mathcal{E}_{p,q}(\psi) := \left\{ \begin{pmatrix} \xi \\ y \end{pmatrix} \in \mathbb{R}^2; \quad |q\xi + p - y| \leq \psi(q) \right\},$$

and for every positive integer q, let

$$\mathcal{E}_q(\psi) := \bigcup_{\substack{p \in \mathbb{Z} \\ \gcd(p,q)=1}} \mathcal{E}_{p,q}(\psi)$$

be the union of all relevant strips involved in (8) for fixed q. Without loss of generality, we shall assume that $\psi(q) \leq 1/2$, so that the above union is disjoint. Then $\mathcal{E}(\psi)$ is equal to the lim sup set

$$\mathcal{E}(\psi) = \bigcap_{Q \geq 1} \bigcup_{q \geq Q} \mathcal{E}_q(\psi).$$

As usual when dealing with lim sup set in metrical theory, we first estimate Lebesgue measure of pairwise intersections of the subsets $\mathcal{E}_q(\psi), q \geq 1$. We establish next a new kind of zero-one law.

3.1 Measuring intersections

In this section, we restrict our attention to points located in the unit square $[0, 1]^2$. We denote by φ the Euler totient function and by λ the Lebesgue measure on $\mathbb{R}^2$.

Lemma 2. *Let $\psi : \mathbb{N} \to [0, 1/2]$ be a function.*
(i) *For every positive integer q, we have*

$$\lambda(\mathcal{E}_q(\psi) \cap [0, 1]^2) = \frac{2\varphi(q)\psi(q)}{q}.$$

(ii) *Let q and s be distinct positive integers. Then, we have the upper bound*

$$\lambda(\mathcal{E}_q(\psi) \cap \mathcal{E}_s(\psi) \cap [0, 1]^2) \leq 4\psi(q)\psi(s).$$

Proof. Denote by χ_q the characteristic function of the interval $[-\psi(q), \psi(q)]$. Then the characteristic function $\chi_{\mathcal{E}_q(\psi)}$ of the subset $\mathcal{E}_q(\psi) \subset \mathbb{R}^2$ is equal to

$$\chi_{\mathcal{E}_q(\psi)}(\xi, y) = \sum_{\substack{p\in\mathbb{Z}\\ \gcd(p,q)=1}} \chi_q(q\xi + p - y) = \sum_{\substack{p\in\mathbb{Z}\\ \gcd(p,q)=1}} \chi_q(q\xi - p - y).$$

Observe that if $\begin{pmatrix}\xi\\ y\end{pmatrix}$ belongs to $[0,1]^2$, the indices p of non-vanishing terms occurring in the last sum are located in the interval $-1 \leq p \leq q$. Integrating first with respect to x, we find

$$\begin{aligned}
\lambda(\mathcal{E}_q(\psi) \cap [0,1]^2) &= \int_0^1 \int_0^1 \chi_{\mathcal{E}_q(\psi)}(x, y) dx dy \\
&= \sum_{\substack{p\in\mathbb{Z}\\ -1\leq p\leq q,\, \gcd(p,q)=1}} \int_0^1 \int_0^1 \chi_q(qx - p - y) dx dy \\
&= \int_{1-\psi(q)}^1 \frac{-1 + y + \psi(q)}{q} dy + \sum_{\substack{1\leq p\leq q-2\\ \gcd(p,q)=1}} \int_0^1 \frac{2\psi(q)}{q} dy \\
&\qquad + \int_0^{1-\psi(q)} \frac{2\psi(q)}{q} dy + \int_{1-\psi(q)}^1 \frac{1 - y + \psi(q)}{q} dy \\
&= \frac{2\varphi(q)\psi(q)}{q}.
\end{aligned}$$

The first term appearing in the third equality of the above formula corresponds to the summation index $p = -1$ and the two last ones to $p = q - 1$. We have thus proved (i).

For the second assertion, we majorize

$$\begin{aligned}
\lambda(\mathcal{E}_q(\psi) \cap \mathcal{E}_s(\psi) \cap [0,1]^2) &= \int_0^1 \int_0^1 \chi_{\mathcal{E}_q(\psi)}(x, y) \chi_{\mathcal{E}_s(\psi)}(x, y) dx dy \\
&\leq \int_0^1 \int_0^1 \left(\sum_{p\in\mathbb{Z}} \chi_q(qx + p - y) \right) \left(\sum_{r\in\mathbb{Z}} \chi_s(sx + r - y) \right) dx dy \\
&= \int_0^1 \int_0^1 \chi_q(\|qx - y\|) \chi_s(\|sx - y\|) dx dy,
\end{aligned}$$

where $\|.\|$ stands as usual for the distance to the nearest integer. Now, (ii) follows from the probabilistic independence formula

$$\int_0^1 \int_0^1 \chi_q(\|qx - y\|) \chi_s(\|sx - y\|) dx dy = 4\psi(q)\psi(s),$$

obtained by Cassels on page 124 of [4] (see Proof (*ii*)). $\square$

3.2 A zero-one law

We say that a subset of $\mathbb{R}^2$ is a *null* set if it has Lebesgue measure 0. A set whose complementary is a null set is called a *full* set. The goal of this section is to prove the

Proposition. *Let ψ be an approximating function as in Theorem 2. Then the subset $\mathcal{E}(\psi)$ is either a null set or a full set.*

For proving the proposition, it is convenient to introduce the larger subset

$$\mathcal{E}'(\psi) = \bigcup_{k \geq 1} \mathcal{E}(k\psi).$$

In other words, $\mathcal{E}'(\psi)$ is the set of all points $\begin{pmatrix} \xi \\ y \end{pmatrix}$ in $\mathbb{R}^2$ for which there exist a positive real number κ, depending possibly on $\begin{pmatrix} \xi \\ y \end{pmatrix}$, and infinitely many primitive points (p, q) satisfying

$$q \geq 1 \quad \text{and} \quad |q\xi + p - y| \leq \kappa\psi(q). \tag{9}$$

Observe that $\mathcal{E}(k\psi) \subseteq \mathcal{E}(k'\psi)$ if $1 \leq k \leq k'$. In particular, $\mathcal{E}(\psi)$ is contained in $\mathcal{E}'(\psi)$.

Lemma 3. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function tending to zero at infinity. Then the difference $\mathcal{E}'(\psi) \setminus \mathcal{E}(\psi)$ is a set of null Lebesgue measure.*

Proof. We show that all sets $\mathcal{E}(k\psi), k \geq 1$, have the same Lebesgue measure. For every real number y, denote by $\mathcal{E}(\psi, y) \subseteq \mathbb{R}$ the section of $\mathcal{E}(\psi)$ on the horizontal line $\mathbb{R} \times \{y\}$, i.e.

$$\mathcal{E}(\psi, y) = \left\{ \xi \in \mathbb{R}\,;\, \begin{pmatrix} \xi \\ y \end{pmatrix} \in \mathcal{E}(\psi) \right\}.$$

Then, using (8), we can express

$$\mathcal{E}(\psi, y) = \bigcap_{Q \geq 1} \bigcup_{q \geq Q} \bigcup_{\substack{p \in \mathbb{Z} \\ \gcd(p,q)=1}} \left[\frac{-p + y - \psi(q)}{q}, \frac{-p + y + \psi(q)}{q} \right]$$

as a limsup set of intervals. If we restrict to a bounded part of $\mathcal{E}(\psi, y)$, the above union over p reduces to a finite one. Observe that the centers $\frac{-p+y}{q}$ of these intervals do not depend on ψ, and that their length is multiplied by the constant factor k when replacing ψ by $k\psi$. Appealing now to a result due to Cassels [5], we infer that all $\limsup$ sets $\mathcal{E}(k\psi, y), k \geq 1$, have the same Lebesgue measure. See also Corollary of Lemma 2.1 on page 30 of [8]. Notice that for fixed k, the length $\frac{2k\psi(q)}{q}$ of the intervals

$\left[\frac{-p+y-k\psi(q)}{q}, \frac{-p+y+k\psi(q)}{q}\right]$ tend to 0 as q tends to infinity, as required by Lemma 2.1. By Fubini, the fibered sets

$$\mathcal{E}(k\psi) = \coprod_{y\in\mathbb{R}} \Big(\mathcal{E}(k\psi, y) \times \{y\}\Big), \quad k \geq 1,$$

have as well the same Lebesgue measure in $\mathbb{R}^2$. $\square$

Lemma 4. *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a non-increasing function satisfying* (3)*. Then $\mathcal{E}'(\psi)$ is either a null or a full set.*

Proof. It is based on the following observation. Let $\begin{pmatrix} \xi \\ y \end{pmatrix}$ belong to $\mathcal{E}'(\psi)$ and let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $SL(2,\mathbb{Z})$ such that $c\xi + d > 0$. Then the point $\begin{pmatrix} \xi' \\ y' \end{pmatrix}$ with coordinates

$$\xi' = \frac{a\xi + b}{c\xi + d} \quad \text{and} \quad y' = \frac{y}{c\xi + d}$$

belongs to $\mathcal{E}'(\psi)$. Indeed, substituting

$$q = aq' + cp', \quad p = bq' + dp' \tag{10}$$

in (9) and dividing by $c\xi + d$, we obtain the inequalities

$$q' \geq 1 \quad \text{and} \quad |q'\xi' + p' - y'| \leq \frac{\kappa}{c\xi + d}\psi(q) \leq \kappa'\psi(q'), \tag{11}$$

for some $\kappa' > 0$ independent of q'. The positivity of q' is proved as follows. Note that (9) implies the estimate

$$p = -q\xi + \mathcal{O}_{\xi,y}(1).$$

Then, inverting the linear substitution (10), we find

$$q' = dq - cp = q(c\xi + d) + \mathcal{O}_{\gamma,\xi,y}(1).$$

Since we have assumed that $c\xi + d > 0$, the term $q(c\xi + d)$ is arbitrarily large when q is large enough. The condition (3) now shows that $\psi(q) \asymp \psi(q')$. Thus (11) is satisfied for infinitely many primitive points (p', q'), since the linear substitution (10) is unimodular. We have shown that $\begin{pmatrix} \xi' \\ y' \end{pmatrix}$ belongs to $\mathcal{E}'(\psi)$.

We now prove that the intersection $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$ is either a full or a null subset of the half plane $\mathbb{R} \times \mathbb{R}^+$. To that purpose, we consider the map

$$\Phi : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+, \quad \text{defined by} \quad \Phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x/y \\ 1/y \end{pmatrix}.$$

Clearly Φ is a continuous involution of $\mathbb{R} \times \mathbb{R}^+$. The image

$$\Omega := \Phi\Big(\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)\Big)$$

is formed by all points of the type

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\xi}{y} \\ \frac{1}{y} \end{pmatrix},$$

where $\begin{pmatrix} \xi \\ y \end{pmatrix}$ ranges over $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$. Now, the above condition $c\xi + d > 0$ is obviously equivalent to $cu + dv > 0$ since y is positive. Then, the point

$$\Phi \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix} = \begin{pmatrix} \frac{au+bv}{cu+dv} \\ \frac{1}{cu+dv} \end{pmatrix} = \begin{pmatrix} \frac{a\xi+b}{c\xi+d} \\ \frac{y}{c\xi+d} \end{pmatrix}$$

belongs to $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$, by the preceding observation. Applying the involution Φ, we find that

$$\Phi\left(\begin{pmatrix} \frac{a\xi+b}{c\xi+d} \\ \frac{y}{c\xi+d} \end{pmatrix}\right) = \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to Ω. In other words, setting $\Gamma = SL(2, \mathbb{Z})$, we have established the inclusion

$$(\Gamma\Omega) \cap (\mathbb{R} \times \mathbb{R}^+) \subseteq \Omega.$$

Since the reversed inclusion is obvious, the equality $\Omega = (\Gamma\Omega) \cap (\mathbb{R} \times \mathbb{R}^+)$ holds in fact. Assuming that Ω is not a null set, the ergodicity of the linear action of Γ on $\mathbb{R}^2$ [13] shows that $\Gamma\Omega$ is a full set in $\mathbb{R}^2$. Hence Ω is a full set in the half plane $\mathbb{R} \times \mathbb{R}^+$. Transforming now Ω by Φ, we find that

$$\Phi(\Omega) = \mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+),$$

is as well a full set in $\mathbb{R} \times \mathbb{R}^+$, thus proving the claim.

We finally use another transformation to carry the zero-one law from the positive half plane $\mathbb{R} \times \mathbb{R}^+$ to the negative one $\mathbb{R} \times \mathbb{R}^-$. Writing (9) in the equivalent form

$$q \geq 1 \quad \text{and} \quad |q(-\xi) + (-p) - (-y)| \leq \kappa\psi(q),$$

shows that $\mathcal{E}'(\psi)$ is invariant under the symmetry $\begin{pmatrix} \xi \\ y \end{pmatrix} \mapsto \begin{pmatrix} -\xi \\ -y \end{pmatrix}$ which maps $\mathbb{R} \times \mathbb{R}^+$ onto $\mathbb{R} \times \mathbb{R}^-$. Therefore $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^-)$ is a null or a full set in $\mathbb{R} \times \mathbb{R}^-$ when $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$ is accordingly a null or a full set in $\mathbb{R} \times \mathbb{R}^+$. □

Now, the combination of Lemma 3 and Lemma 4 obviously yields our proposition.

3.3 Concluding the proof of Theorem 2

Assume first that $\sum \psi(\ell)$ converges. We have to show that the set

$$\mathcal{E}(\psi) = \limsup_{q \to +\infty} \mathcal{E}_q(\psi)$$

has null Lebesgue measure. Lemma 2 shows that the partial sums

$$\sum_{q=1}^{Q} \lambda(\mathcal{E}_q(\psi) \cap [0,1]^2) = 2 \sum_{q=1}^{Q} \frac{\varphi(q)\psi(q)}{q} \leq 2 \sum_{q=1}^{Q} \psi(q)$$

converge (*). Then, Borel-Cantelli Lemma ensures that the lim sup set $\mathcal{E}(\psi) \cap [0,1]^2$ is a null set. Thus $\mathcal{E}(\psi)$ cannot be a full set. Now, the above proposition tells us that $\mathcal{E}(\psi)$ is a null set.

We now consider the case of a divergent series $\sum \psi(\ell)$. Observe that the estimate

$$\frac{1}{2} \sum_{q=1}^{Q} \psi(q) \leq \sum_{q=1}^{Q} \frac{\varphi(q)\psi(q)}{q} \leq \sum_{q=1}^{Q} \psi(q) \tag{12}$$

holds true for any large integer Q, since the sequence $\psi(\ell)_{\ell \geq 1}$ is non-increasing. The right inequality is obvious, while the left one easily follows from Abel summation process. See for instance Chapter 2 of [8], where full details are provided. By Lemma 2 and (12), the sums

$$\sum_{q=1}^{Q} \lambda(\mathcal{E}_q(\psi) \cap [0,1]^2) = 2 \sum_{q=1}^{Q} \frac{\varphi(q)\psi(q)}{q} \geq \sum_{q=1}^{Q} \psi(q)$$

are then unbounded. Then, using a classical converse to Borel-Cantelli Lemma, we have the lower bound

$$\begin{aligned}
\lambda\Big(\mathcal{E}(\psi) \cap [0,1]^2\Big) &= \lambda\Big(\limsup_{q \to +\infty} (\mathcal{E}_q(\psi) \cap [0,1]^2)\Big) \\
&\geq \limsup_{Q \to +\infty} \frac{\Big(\sum_{q=1}^{Q} \lambda(\mathcal{E}_q(\psi) \cap [0,1]^2)\Big)^2}{\sum_{q=1}^{Q} \sum_{s=1}^{Q} \lambda(\mathcal{E}_q(\psi) \cap \mathcal{E}_s(\psi) \cap [0,1]^2)}.
\end{aligned} \tag{13}$$

See for instance Lemma 2.3 in [8]. Lemma 2 and (12) now show that the numerator on the right hand side of (13) equals

$$4 \left(\sum_{q=1}^{Q} \frac{\varphi(q)\psi(q)}{q} \right)^2 \geq \left(\sum_{q=1}^{Q} \psi(q) \right)^2,$$

(*) Here again we assume without loss of generality that $\psi(q) \leq 1/2$ for every $q \geq 1$, so that Lemma 2 may be applied.

when Q is large, while the denominator is bounded from above by

$$4\sum_{\substack{q=1,s=1\\q\neq s}}^{Q}\psi(q)\psi(s)+2\sum_{q=1}^{Q}\psi(q)\leq 4\left(\sum_{q=1}^{Q}\psi(q)\right)^2+2\sum_{q=1}^{Q}\psi(q).$$

Thus (13) yields the lower bound

$$\lambda\Big(\mathcal{E}(\psi)\cap[0,1]^2\Big)\geq\frac{1}{4}.$$

Hence $\mathcal{E}(\psi)$ is not a null set; it is thus a full set according to our proposition.

4 An approach to our problem

In this section, we apply a transference principle between homogeneous and inhomogeneous approximation, as displayed in Chapter V of [4] and in [3], for constructing explicit integer solutions of the inequality

$$|q\xi+p-y|\leq\frac{2}{|q|}. \tag{14}$$

Let $(p_k/q_k)_{k\geq 0}$ be the sequence of convergents to the irrational number ξ. The theory of continued fractions, see for instance the monograph [9], tells us that

$$|q_k\xi-p_k|\leq\frac{1}{q_{k+1}}\quad\text{and}\quad p_kq_{k+1}-p_{k+1}q_k=(-1)^{k+1}, \tag{15}$$

for any $k\geq 0$. Setting $\nu_k=(-1)^{k+1}q_ky$, we thus have the relations

$$\nu_kq_{k+1}+\nu_{k+1}q_k=0\quad\text{and}\quad\nu_k(q_{k+1}\xi-p_{k+1})+\nu_{k+1}(q_k\xi-p_k)=y. \tag{16}$$

Now, let n_k be anyone of the two integers $\lfloor\nu_k\rfloor$ and $\lceil\nu_k\rceil$ (†). Then,

$$|\nu_k-n_k|<1, \tag{17}$$

and n_k is either equal to $(-1)^{k+1}\lfloor yq_k\rfloor$ or to $(-1)^{k+1}\lceil yq_k\rceil$. Setting

$$p=-n_kp_{k+1}-n_{k+1}p_k\quad\text{and}\quad q=n_kq_{k+1}+n_{k+1}q_k, \tag{18}$$

(†) As usual $\lfloor x\rfloor$ and $\lceil x\rceil$ stand respectively for the floor and the ceiling of the real number x. Then $\lceil x\rceil=\lfloor x\rfloor+1$, unless x is an integer in which case $\lfloor x\rfloor=\lceil x\rceil=x$.

we deduce from (16) the expressions

$$
\begin{aligned}
q\xi + p - y &= n_k(q_{k+1}\xi - p_{k+1}) + n_{k+1}(q_k\xi - p_k) - y \\
&= (n_k - \nu_k)(q_{k+1}\xi - p_{k+1}) + (n_{k+1} - \nu_{k+1})(q_k\xi - p_k)
\end{aligned}
\tag{19}
$$

and

$$
q = (n_k - \nu_k)q_{k+1} + (n_{k+1} - \nu_{k+1})q_k. \tag{20}
$$

Recall that $q_k\xi - p_k$ and $q_{k+1}\xi - p_{k+1}$ have opposite signs. Assuming that $n_k - \nu_k$ and $n_{k+1} - \nu_{k+1}$ have the same sign, we infer from the formulas (19), (20) and from (15), (17) that

$$
|q\xi + p - y| < \frac{1}{q_{k+1}} \quad \text{and} \quad |q| < 2q_{k+1}. \tag{21}
$$

Otherwise, we have

$$
|q\xi + p - y| < \frac{2}{q_{k+1}} \quad \text{and} \quad |q| < q_{k+1}. \tag{22}
$$

The inequalities (21) and (22) obviously imply (14).

Since the linear substitution (18) is unimodular, the integers p and q are coprime if and only if n_k and n_{k+1} are coprime. Recall that the two choices $n_k = \lfloor \nu_k \rfloor$ and $n_k = \lceil \nu_k \rceil$ are admissible, both for n_k and n_{k+1}. It thus remains to find indices k for which at least one of the coprimality conditions

$$
\begin{aligned}
&\gcd(\lfloor yq_k \rfloor, \lfloor yq_{k+1} \rfloor) = 1 \quad \text{or} \quad \gcd(\lceil yq_k \rceil, \lceil yq_{k+1} \rceil) = 1 \\
\text{or} \quad &\gcd(\lfloor yq_k \rfloor, \lceil yq_{k+1} \rceil) = 1 \quad \text{or} \quad \gcd(\lceil yq_k \rceil, \lfloor yq_{k+1} \rfloor) = 1,
\end{aligned}
\tag{23}
$$

is verified. Note that (23) obviously fails for all $k \geq 0$ when y is an integer not equal to 1 or to -1. Otherwise, the contingent existence of infinitely many indices k satisfying (23) is a non-trivial problem that we leave hanging.

Let us mention that the proof of (1) in [6] follows the same idea, finding a primitive integer point inside the square centered at the point $(\nu_k, \nu_{k+1}) \in \mathbb{R}^2$ with side $C \log|\nu_k| / \log\log|\nu_k|$ for some suitable large absolute constant C.

4.1 Proof of Theorem 3

We quote the following metrical result due to Harman (Theorem 8.3 in [8]). Assume that the series (5) diverges. Then for almost all positive real numbers y, there exist infinitely many indices k such that the integer part $\lfloor yq_k \rfloor$ is a prime number. These indices k fulfill (23) since, assuming for simplicity that y is irrational, either $\lfloor yq_{k+1} \rfloor$ or $\lceil yq_{k+1} \rceil = \lfloor yq_{k+1} \rfloor + 1$ is not divisible by $\lfloor yq_k \rfloor$ and is thus relatively prime with $\lfloor yq_k \rfloor$. Hence (14) has infinitely many coprime solutions (p, q) for almost every positive real number y. Writing now (14) in the equivalent form

$$|(-q)\xi + (-p) - (-y)| \leq \frac{2}{|q|}$$

shows that, ξ being given, the set of all real numbers y for which (14) has infinitely many coprime solutions is invariant by the symmetry $y \mapsto -y$. The first assertion is thus established. To complete the proof, note that

$$\lim_{k \to +\infty} \frac{\log q_k}{k} = \frac{\pi^2}{12 \log 2}$$

for almost every ξ by Khintchine-Levy Theorem (see equation (4.18) in [2]). Thus the series (5) diverges for almost every ξ.

5 Generic density exponents

We prove in this section Theorem 4, as a consequence of Borel-Cantelli Lemma combined with the following counting result.

Lemma 5. *Let $\mathbf{x}$ be a point in $\mathbb{R}^2$ whose orbit $\Gamma\mathbf{x}$ is dense in $\mathbb{R}^2$. For every symmetric compact set Ω in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ there exists $c > 0$ such that*

$$\mathrm{Card}\{\gamma \in \Gamma; \gamma\mathbf{x} \in \Omega, |\gamma| \leq T\} \leq cT$$

for any real number $T \geq 1$.

Proof. Ledrappier [11] has shown that the limit formula

$$\lim_{T \to +\infty} \frac{1}{T} \sum_{\gamma \in \Gamma, |\gamma| \leq T} f(\gamma\mathbf{x}) = \frac{4}{|\mathbf{x}| \mathrm{vol}(\Gamma \setminus SL(2, \mathbb{R}))} \int \frac{f(\mathbf{y})}{|\mathbf{y}|} d\mathbf{y}$$

holds for any even continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ having compact support on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, with a suitable normalisation of Haar measure on $SL(2, \mathbb{R})$. Approximating uniformly from above and from below the characteristic function of Ω by even continuous functions, we deduce that

$$\lim_{T \to +\infty} \frac{\mathrm{Card}\{\gamma \in \Gamma; \gamma\mathbf{x} \in \Omega, |\gamma| \leq T\}}{T} = \frac{4}{|\mathbf{x}| \mathrm{vol}(\Gamma \setminus SL(2, \mathbb{R}))} \int_\Omega \frac{d\mathbf{y}}{|\mathbf{y}|}.$$

Lemma 5 immediately follows. $\square$

For any point $\mathbf{y} \in \mathbb{R}^2$ and any positive real number r, we denote by

$$B(\mathbf{y}, r) = \{\mathbf{z} \in \mathbb{R}^2; |\mathbf{z} - \mathbf{y}| \leq r\}$$

the closed disc centered at $\mathbf{y}$ with radius r.

Lemma 6. *Let $\mathbf{x}$ be a point in $\mathbb{R}^2$ whose orbit $\Gamma\mathbf{x}$ is dense, Ω a symmetric compact set in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ and μ a real number $> 1/2$. For every integer $n \geq 1$, put*

$$\mathcal{B}_n = \bigcup_{\substack{\gamma \in \Gamma \\ |\gamma| = n, \gamma\mathbf{x} \in \Omega}} B(\gamma\mathbf{x}, n^{-\mu}).$$

Then the set

$$\mathcal{B} := \limsup_{n \to +\infty} \mathcal{B}_n = \bigcap_{N \geq 1} \bigcup_{n \geq N} \mathcal{B}_n = \bigcap_{N \geq 1} \bigcup_{\substack{\gamma \in \Gamma \\ |\gamma| \geq N, \gamma\mathbf{x} \in \Omega}} B(\gamma\mathbf{x}, |\gamma|^{-\mu})$$

has null Lebesgue measure.

Proof. We apply Borel-Cantelli Lemma and we prove that the series $\sum_{n \geq 1} \lambda(\mathcal{B}_n)$ converges if $\mu > 1/2$.

For every positive integer n, set

$$M_n = \mathrm{Card}\{\gamma \in \Gamma; \gamma\mathbf{x} \in \Omega, |\gamma| = n\}.$$

Lemma 5 gives us the upper bound

$$M_1 + \cdots + M_n = \mathrm{Card}\{\gamma \in \Gamma; \gamma\mathbf{x} \in \Omega, |\gamma| \leq n\} \leq cn, \tag{24}$$

for some $c > 0$ independent of $n \geq 1$. Since a ball of radius r has Lebesgue measure $4r^2$, we trivially bound from above

$$\lambda(\mathcal{B}_n) \leq \sum_{\substack{\gamma \in \Gamma \\ |\gamma| = n, \gamma\mathbf{x} \in \Omega}} 4n^{-2\mu} = 4M_n n^{-2\mu}.$$

Summing by parts, we deduce from (24) that

$$\begin{aligned} \sum_{n=1}^{N} \frac{M_n}{n^{2\mu}} &= \sum_{n=1}^{N-1} (M_1 + \cdots + M_n) \left(\frac{1}{n^{2\mu}} - \frac{1}{(n+1)^{2\mu}} \right) + \frac{M_1 + \cdots + M_N}{N^{2\mu}} \\ &\leq c \sum_{n=1}^{N-1} n \left(\frac{1}{n^{2\mu}} - \frac{1}{(n+1)^{2\mu}} \right) + \frac{cN}{N^{2\mu}} = c \sum_{n=1}^{N} \frac{1}{n^{2\mu}}. \end{aligned}$$

The partial sums

$$\sum_{n=1}^{N} \lambda(\mathcal{B}_n) \le 4 \sum_{n=1}^{N} \frac{M_n}{n^{2\mu}} \le 4c \sum_{n=1}^{N} \frac{1}{n^{2\mu}}$$

thus converge if $\mu > 1/2$. $\square$

5.1 Proof of Theorem 4

We argue by contradiction and suppose on the contrary that $\mu_\Gamma(\mathbf{x}) > 1/2$. Fix a real number μ with $1/2 < \mu < \mu_\Gamma(\mathbf{x})$. Then for almost all points $\mathbf{y} \in \mathbb{R}^2$, we have $\mu(\mathbf{x}, \mathbf{y}) > \mu$. This means that there exist infinitely many $\gamma \in \Gamma$ satisfying (6), or equivalently that $\mathbf{y}$ belongs to infinitely many balls of the form $B(\gamma\mathbf{x}, |\gamma|^{-\mu})$. We now restrict our attention to points $\mathbf{y}$ with $\mu(\mathbf{x}, \mathbf{y}) > \mu$ lying in an annulus

$$\Omega' = \{\mathbf{z} \in \mathbb{R}^2; a' \le |\mathbf{z}| \le b'\},$$

where $b' > a' > 0$ are arbitrarily fixed. Since $\mathbf{y}$ belongs to the intersection $\Omega' \cap B(\gamma\mathbf{x}, |\gamma|^{-\mu})$, we deduce from the triangle inequality the estimate

$$a' - |\gamma|^{-\mu} \le |\gamma\mathbf{x}| \le b' + |\gamma|^{-\mu}.$$

Fixing $a < a'$ and $b > b'$, the center $\gamma\mathbf{x}$ then lies in the larger annulus

$$\Omega = \{\mathbf{z} \in \mathbb{R}^2; a \le |\mathbf{z}| \le b\},$$

provided that $|\gamma|$ is large enough. It follows that $\mathbf{y}$ falls inside the union of balls

$$\bigcup_{\substack{\gamma\in\Gamma \\ |\gamma|\ge N, \gamma\mathbf{x}\in\Omega}} B(\gamma\mathbf{x}, |\gamma|^{-\mu})$$

considered in Lemma 6 for every integer N large enough, and thus $\mathbf{y}$ belongs to $\mathcal{B}$. However, Lemma 6 asserts that $\mathcal{B}$ is a null set which is a contradiction.

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